TP2: TÍCH PHÂN HÀM SỐ VÔ TỈ

Dạng 1: Đổi biến số dạng 1

Câu 1. $I=\int \frac{x}{3x+\sqrt{9x^2-1}}dx$

• $I=\int \frac{x}{3x+\sqrt{9x^2-1}}dx=\int x(3x-\sqrt{9x^2-1})dx=\int 3x^2dx-\int x\sqrt{9x^2-1}dx$

$+ I_1=\int 3x^2dx=x^3+C_1 \quad + I_2=\int x\sqrt{9x^2-1}dx=\frac{1}{18}\int \sqrt{9x^2-1}\,d(9x^2-1)=\frac{1}{27}(9x^2-1)^{\frac{3}{2}}+C_2$

$\Rightarrow I=\frac{1}{27}(9x^2-1)^{\frac{3}{2}}+x^3+C$

Câu 2. $I=\int \frac{x^2+\sqrt{x}}{\sqrt{1+x\sqrt{x}}}dx$

• $\int \frac{x^2+\sqrt{x}}{\sqrt{1+x\sqrt{x}}}dx=\int \frac{x^2}{\sqrt{1+x\sqrt{x}}}dx+\int \frac{\sqrt{x}}{\sqrt{1+x\sqrt{x}}}dx$.

$+ I_1=\int \frac{x^2}{\sqrt{1+x\sqrt{x}}}dx$. *Đặt* $t=\sqrt{1+x\sqrt{x}} \Leftrightarrow t^2-1=x\sqrt{x} \Leftrightarrow x^3=(t^2-1)^2 \Leftrightarrow x^2dx=\frac{4}{3}t(t^2-1)dt$

$\Rightarrow \int \frac{4}{3}(t^2-1)dt=\frac{4}{9}t^3-\frac{4}{3}t+C = \frac{4}{9}\left(\sqrt{1+x\sqrt{x}}\right)^3-\frac{4}{3}\sqrt{1+x\sqrt{x}}+C_1$

$+ I_2=\int \frac{\sqrt{x}}{\sqrt{1+x\sqrt{x}}}dx = \frac{2}{3}\int \frac{d(1+x\sqrt{x})}{\sqrt{1+x\sqrt{x}}} = \frac{4}{3}\sqrt{1+x\sqrt{x}}+C_2$

Vậy: $I=\frac{4}{9}\left(\sqrt{1+x\sqrt{x}}\right)^3+C$

Câu 3. $I=\int_0^4 \frac{\sqrt{2x+1}}{1+\sqrt{2x+1}}dx$ • *Đặt* $t=\sqrt{2x+1}$. $I=\int_1^3 \frac{t^2}{1+t}dt=2+\ln 2$.

Câu 4. $I=\int_2^6 \frac{dx}{2x+1+\sqrt{4x+1}}$ • *Đặt* $t=\sqrt{4x+1}$. $I=\ln\frac{3}{2}-\frac{1}{12}$

Câu 5. $I=\int_0^1 x^3\sqrt{1-x^2}dx$ • *Đặt:* $t=\sqrt{1-x^2} \Rightarrow I=\int_0^1 \left(t^2-t^4\right)dt=\frac{2}{15}$.

Câu 6. $I=\int_0^1 \frac{1+x}{1+\sqrt{x}}dx$

• *Đặt* $t=\sqrt{x} \Rightarrow dx=2t.dt$. $I=2\int_0^1 \frac{t^3+t}{t+1}dt = 2\int_0^1 \left(t^2-t+2-\frac{2}{1+t}\right)dt = \frac{11}{3}-4\ln 2$.

Câu 7. $I=\int_0^3 \frac{x-3}{3\sqrt{x+1}+x+3}dx$

• *Đặt* $t=\sqrt{x+1}\Rightarrow 2tdu=dx \ \Rightarrow I=\int_1^2\frac{2t^3-8t}{t^2+3t+2}dt=\int_1^2(2t-6)dt+6\int_1^2\frac{1}{t+1}dt=-3+6\ln\frac{3}{2}$

Câu 8. $I=\int_{-1}^{0}x.\sqrt[3]{x+1}dx$

• *Đặt* $t=\sqrt[3]{x+1}\Rightarrow t^3=x+1\Rightarrow dx=3t^2dt\Rightarrow I=\int_0^1 3(t^3-1)dt=3\left(\frac{t^7}{7}-\frac{t^4}{4}\right)\Big|_0^1=-\frac{9}{28}$

Câu 9. $I=\int_1^5\frac{x^2+1}{x\sqrt{3x+1}}dx$

• *Đặt* $t=\sqrt{3x+1}\ \Rightarrow dx=\frac{2tdt}{3}\ \Rightarrow I=\int_2^4\frac{\left(\frac{t^2-1}{3}\right)^2+1}{\frac{t^2-1}{3}.t}.\frac{2tdt}{3}\ =\frac{2}{9}\int_2^4(t^2-1)dt+2\int_2^4\frac{dt}{t^2-1}$

$=\frac{2}{9}\left(\frac{1}{3}t^3-t\right)\Big|_2^4+\ln\left|\frac{t-1}{t+1}\right|\Big|_2^4=\frac{100}{27}+\ln\frac{9}{5}.$

Câu 10. $I=\int_0^3\frac{2x^2+x-1}{\sqrt{x+1}}dx$

• *Đặt* $\sqrt{x+1}=t\Leftrightarrow x=t^2-1\ \Rightarrow dx=2tdt$

$\Rightarrow I=\int_1^2\frac{2(t^2-1)^2+(t^2-1)-1}{t}2tdt\ =2\int_1^2(2t^4-3t^2)dt=\left(\frac{4t^5}{5}-2t^3\right)\Big|_1^2=\frac{54}{5}$

Câu 11. $I=2\int_0^1\frac{x^2dx}{(x+1)\sqrt{x+1}}$

• *Đặt* $t=\sqrt{x+1}\Rightarrow t^2=x+1\Rightarrow 2tdt=dx$

$\Rightarrow I=\int_1^{\sqrt{2}}\frac{(t^2-1)^2}{t^3}.2tdt=2\int_1^{\sqrt{2}}\left(t-\frac{1}{t}\right)^2dt=2\left(\frac{t^3}{3}-2t-\frac{1}{t}\right)\Big|_1^{\sqrt{2}}=\frac{16-11\sqrt{2}}{3}$

Câu 12. $I=\int_0^4\frac{x+1}{\left(1+\sqrt{1+2x}\right)^2}dx$

• *Đặt* $t=1+\sqrt{1+2x}\Rightarrow dt=\frac{dx}{\sqrt{1+2x}}\Rightarrow dx=(t-1)dt$ *và* $x=\frac{t^2-2t}{2}$

Ta có: $I=\frac{1}{2}\int_2^4\frac{(t^2-2t+2)(t-1)}{t^2}dt=\frac{1}{2}\int_2^4\frac{t^3-3t^2+4t-2}{t^2}dt=\frac{1}{2}\int_2^4\left(t-3+\frac{4}{t}-\frac{2}{t^2}\right)dt$

$=\frac{1}{2}\left(\frac{t^2}{2}-3t+4\ln|t|+\frac{2}{t}\right)\Big|=2\ln 2-\frac{1}{4}$

Câu 13. $I=\int_{\sqrt{3}}^{\sqrt{8}}\frac{x-1}{\sqrt{x^2+1}}dx$

• $I=\int_{\sqrt{3}}^{\sqrt{8}}\left(\frac{x}{\sqrt{x^2+1}}-\frac{1}{\sqrt{x^2+1}}\right)dx=\left[\sqrt{x^2+1}-\ln\left(x+\sqrt{x^2+1}\right)\right]\Big|_{\sqrt{3}}^{\sqrt{8}}=1+\ln\left(\sqrt{3}+2\right)-\ln\left(\sqrt{8}+3\right)$

Câu 14. $I=\int_0^1 (x-1)^3\sqrt{2x-x^2}\,dx$

• $I=\int_0^1 (x-1)^3\sqrt{2x-x^2}\,dx=\int_0^1 (x^2-2x+1)\sqrt{2x-x^2}\,(x-1)dx$. *Đặt* $t=\sqrt{2x-x^2}\Rightarrow I=-\frac{2}{15}$.

Câu 15. $I=\int_0^2 \frac{2x^3-3x^2+x}{\sqrt{x^2-x+1}}dx$

• $I=\int_0^2 \frac{(x^2-x)(2x-1)}{\sqrt{x^2-x+1}}dx$. *Đặt* $t=\sqrt{x^2-x+1}\Rightarrow I=2\int_1^{\sqrt{3}}(t^2-1)dt=\frac{4}{3}$.

Câu 16. $I=\int_0^2 \frac{x^3dx}{\sqrt[3]{4+x^2}}$

• *Đặt* $t=\sqrt[3]{4+x^2}\Rightarrow x^2=t^3-4\Rightarrow 2xdx=3t^2dt\Rightarrow I=\frac{3}{2}\int_{\sqrt[3]{4}}^{2}(t^4-4t)dt=-\frac{3}{2}\left(\frac{8}{5}+4\sqrt[3]{2}\right)$

Câu 17. $I=\int_{-1}^{1}\frac{dx}{1+x+\sqrt{1+x^2}}$

• *Ta có:* $I=\int_{-1}^{1}\frac{1+x-\sqrt{1+x^2}}{(1+x)^2-(1+x^2)}dx=\int_{-1}^{1}\frac{1+x-\sqrt{1+x^2}}{2x}dx=\frac{1}{2}\int_{-1}^{1}\left(\frac{1}{x}+1\right)dx-\int_{-1}^{1}\frac{\sqrt{1+x^2}}{2x}dx$

+ $I_1=\frac{1}{2}\int_{-1}^{1}\left(\frac{1}{x}+1\right)dx=\frac{1}{2}\left[\ln|x|+x\right]\Big|_{-1}^{1}=1$

+ $I_2=\int_{-1}^{1}\frac{\sqrt{1+x^2}}{2x}dx$. *Đặt* $t=\sqrt{1+x^2}\Rightarrow t^2=1+x^2\Rightarrow 2tdt=2xdx\Rightarrow I_2=\int_{\sqrt{2}}^{\sqrt{2}}\frac{t^2dt}{2(t^2-1)}=0$

Vậy: $I=1$.

<u>*Cách 2*</u>*: Đặt* $t=x+\sqrt{x^2+1}$.

Câu 18. $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ • *Ta có:* $I=\int_{\frac{1}{3}}^{1}\left(\frac{1}{x^2}-1\right)^{\frac{1}{3}}.\frac{1}{x^3}dx$. *Đặt* $t=\frac{1}{x^2}-1\Rightarrow I=6$.

Câu 19. $I=\int_1^2\frac{\sqrt{4-x^2}}{x}dx$

• *Ta có:* $I=\int_1^2\frac{\sqrt{4-x^2}}{x^2}xdx$. *Đặt* $t=\sqrt{4-x^2}\Rightarrow t^2=4-x^2\Rightarrow tdt=-xdx$

$\Rightarrow I=\int_{\sqrt{3}}^{0}\frac{t(-tdt)}{4-t^2}=\int_{\sqrt{3}}^{0}\frac{t^2}{t^2-4}dt=\int_{\sqrt{3}}^{0}(1+\frac{4}{t^2-4})dt=\left(t+\ln\left|\frac{t-2}{t+2}\right|\right)\Big|_{\sqrt{3}}^{0}=-\left(\sqrt{3}+\ln\left|\frac{2-\sqrt{3}}{2+\sqrt{3}}\right|\right)$

Câu 20. $I=\int_{2}^{2\sqrt{5}}\frac{x}{(x^2+1)\sqrt{x^2+5}}dx$ • *Đặt* $t=\sqrt{x^2+5}\Rightarrow I=\int_{3}^{5}\frac{dt}{t^2-4}=\frac{1}{4}\ln\frac{15}{7}$.

Câu 21. $I=\int_{1}^{27}\frac{\sqrt{x}-2}{x+\sqrt[3]{x^2}}dx$

• *Đặt* $t=\sqrt[6]{x}\Rightarrow I=5\int_{1}^{\sqrt{3}}\frac{t^3-2}{t(t^2+1)}dt=5\int_{1}^{\sqrt{3}}\left[1-\frac{2}{t}+\frac{2t}{t^2+1}-\frac{1}{t^2+1}\right]dt=5\left(\sqrt{3}-1+\ln\frac{2}{3}\right)-\frac{5\pi}{12}$

Câu 22. $I=\int_{0}^{1}\frac{1}{\sqrt{x^2+x+1}}dx$

• *Đặt* $t=x+\sqrt{x^2+x+1}\Rightarrow I=\int_{1}^{1+\sqrt{3}}\frac{2dt}{2t+1}=\ln(2t+1)\Big|_{1}^{1+\sqrt{3}}=\ln\frac{3+2\sqrt{3}}{3}$

Câu 23. $I=\int_{0}^{3}\frac{x^2}{(1+\sqrt{1+x})^2(2+\sqrt{1+x})^2}dx$

• *Đặt* $2+\sqrt{1+x}=t\Rightarrow I=\int_{3}^{4}\left(2t-16+\frac{42}{t}-\frac{36}{t^2}\right)dt=-12+42\ln\frac{4}{3}$

Câu 24. $I=\int_{0}^{3}\frac{x^2}{2(x+1)+2\sqrt{x+1}+x\sqrt{x+1}}dx$

• *Đặt* $t=\sqrt{x+1}\Rightarrow I=\int_{1}^{2}\frac{2t(t^2-1)^2dt}{t(t+1)^2}=2\int_{1}^{2}(t-1)^2dt=\frac{2}{3}(t-1)^3\Big|_{1}^{2}=\frac{2}{3}$

Câu 25. $I=\int_{1}^{2\sqrt{2}}\frac{\sqrt[3]{x-x^3}+2011x}{x^4}dx$

• *Ta có:* $I=\int_{1}^{2\sqrt{2}}\frac{\sqrt[3]{\frac{1}{x^2}-1}}{x^3}dx+\int_{1}^{2\sqrt{2}}\frac{2011}{x^3}dx=M+N$

$M=\int_{1}^{2\sqrt{2}}\frac{\sqrt[3]{\frac{1}{x^2}-1}}{x^3}dx$. *Đặt* $t=\sqrt[3]{\frac{1}{x^2}-1}\Rightarrow M=-\frac{3}{2}\int_{0}^{-\frac{\sqrt[3]{7}}{2}}t^3dt=-\frac{21\sqrt[3]{7}}{128}$

$N=\int_{1}^{2\sqrt{2}}\frac{2011}{x^3}dx=\int_{1}^{2\sqrt{2}}2011x^{-3}dx=\left[-\frac{2011}{2x^2}\right]\Bigg|_{1}^{2\sqrt{2}}=\frac{14077}{16}$

$\Rightarrow I=\frac{14077}{16}-\frac{21\sqrt[3]{7}}{128}$.

Câu 26. $I=\int_{0}^{1}\frac{dx}{(1+x^3).\sqrt[3]{1+x^3}}$

• *Đặt* $t=\sqrt[3]{1+x^3}\Rightarrow I=\int_{1}^{\sqrt[3]{2}}\frac{t^2}{t^4.(t^3-1)^{\frac{2}{3}}}dt=\int_{1}^{\sqrt[3]{2}}\frac{dt}{t^2.(t^3-1)^{\frac{2}{3}}}$

$$= \int_1^{\sqrt[3]{2}} \frac{dt}{t^2.\left[t^3\left(1-\frac{1}{t^3}\right)\right]^{\frac{2}{3}}} = \int_1^{\sqrt[3]{2}} \frac{dt}{t^4\left(1-\frac{1}{t^3}\right)^{\frac{2}{3}}} = \int_1^{\sqrt[3]{2}} \frac{\left(1-\frac{1}{t^3}\right)^{-\frac{2}{3}}}{t^4}dt$$

Đặt $u = 1-\frac{1}{t^3} \Rightarrow du = \frac{3dt}{t^4} \Rightarrow I = \int_0^{\frac{1}{2}} \frac{u^{-\frac{2}{3}}}{3}du = \frac{1}{3}\int_0^{\frac{1}{2}} u^{-\frac{2}{3}}du = \frac{1}{3}\left.\left(\frac{u^{\frac{1}{3}}}{\frac{1}{3}}\right)\right|_0^{\frac{1}{2}} = \left.u^{\frac{1}{3}}\right|_0^{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}}$

Câu 27. $I = \int_{\sqrt{3}}^{2\sqrt{2}} \frac{x^4}{\left(x-\frac{1}{x}\right)\sqrt{x^2+1}}dx$

• *Đặt* $t = \sqrt{x^2+1}$

$\Rightarrow I = \int_2^3 \frac{(t^2-1)^2}{t^2-2}dt = \int_2^3 \frac{t^4-2t^2+1}{t^2-2}dt = \int_2^3 t^2dt + \int_2^3 \frac{1}{t^2-2}dt = \frac{19}{3} + \frac{\sqrt{2}}{4}\ln\left(\frac{4+\sqrt{2}}{4-\sqrt{2}}\right)$

Dạng 2: Đổi biến số dạng 2

Câu 28. $I = \int_0^1 \left(\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} - 2x\ln(1+x)\right)dx$

• *Tính* $H = \int_0^1 \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$. *Đặt* $\sqrt{x} = \cos t; t \in \left[0;\frac{\pi}{2}\right] \Rightarrow H = 2-\frac{\pi}{2}$

• *Tính* $K = \int_0^1 2x\ln(1+x)dx$. *Đặt* $\begin{cases} u = \ln(1+x) \\ dv = 2xdx \end{cases} \Rightarrow K = \frac{1}{2}$

Câu 29. $I = \int_{-2}^{2} (x^5+x^2)\sqrt{4-x^2}dx$

• $I = \int_{-2}^{2} (x^5+x^2)\sqrt{4-x^2}dx = \int_{-2}^{2} x^5\sqrt{4-x^2}dx + \int_{-2}^{2} x^2\sqrt{4-x^2}dx = A + B.$

+ *Tính* $A = \int_{-2}^{2} x^5\sqrt{4-x^2}dx$. *Đặt* $t = -x$. *Tính được:* $A = 0$.

+ *Tính* $B = \int_{-2}^{2} x^2\sqrt{4-x^2}dx$. *Đặt* $x = 2\sin t$. *Tính được:* $B = 2\pi$.

Vậy: $I = 2\pi$.

Câu 30. $I=\int_1^2 \frac{\left(3-\sqrt{4-x^2}\right)dx}{2x^4}$

• *Ta có:* $I=\int_1^2 \frac{3}{2x^4}dx-\int_1^2 \frac{\sqrt{4-x^2}}{2x^4}dx$.

+ *Tính* $I_1=\int_1^2 \frac{3}{2x^4}dx=\frac{3}{2}\int_1^2 x^{-4}dx=\frac{7}{16}$.

+ *Tính* $I_2=\int_1^2 \frac{\sqrt{4-x^2}}{2x^4}dx$. *Đặt* $x=2\sin t\Rightarrow dx=2\cos tdt$.

$\Rightarrow I_2=\frac{1}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2 tdt}{\sin^4 t}=\frac{1}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 t\left(\frac{1}{\sin^2 t}\right)dt=-\frac{1}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 t.d(\cot t)=\frac{\sqrt{3}}{8}$

Vậy: $I=\frac{1}{16}\left(7-2\sqrt{3}\right)$.

Câu 31. $I=\int_0^1 \frac{x^2dx}{\sqrt{4-x^6}}$

• *Đặt* $t=x^3\Rightarrow dt=3x^2dx \Rightarrow I=\frac{1}{3}\int_0^1 \frac{dt}{\sqrt{4-t^2}}$.

Đặt $t=2\sin u,\ u\in\left[0;\frac{\pi}{2}\right]\Rightarrow dt=2\cos udu\Rightarrow I=\frac{1}{3}\int_0^{\frac{\pi}{6}} dt=\frac{\pi}{18}$.

Câu 32. $I=\int_0^2 \sqrt{\frac{2-x}{x+2}}dx$ • *Đặt* $x=2\cos t\Rightarrow dx=-2\sin tdt \Rightarrow I=4\int_0^{\frac{\pi}{2}} \sin^2\frac{t}{2}dt=\pi-2$.

Câu 33. $I=\int_0^1 \frac{x^2dx}{\sqrt{3+2x-x^2}}$

• *Ta có:* $I=\int_0^1 \frac{x^2dx}{\sqrt{2^2-(x-1)^2}}$. *Đặt* $x-1=2\cos t$.

$\Rightarrow I=-\int_{\frac{2\pi}{3}}^{\frac{\pi}{2}} \frac{(1+2\cos t)^2 2\sin t}{\sqrt{4-(2\cos t)^2}}dt=\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\left(3+4\cos t+2\cos 2t\right)dt=\frac{\pi}{2}+\frac{3\sqrt{3}}{2}-4$

Câu 34. $\int_0^{\frac{1}{2}} \sqrt{1-2x\sqrt{1-x^2}}\,dx$ • *Đặt* $x=\sin t \Rightarrow I=\int_0^{\frac{\pi}{6}} (\cos t-\sin t)\cos tdt=\frac{\pi}{12}+\frac{\sqrt{3}}{8}-\frac{1}{8}$

Dạng 3: Tích phân từng phần

Câu 35. $I = \int_{\sqrt{2}}^{3} \sqrt{x^2 - 1}dx$

• *Đặt* $\begin{cases} u = \sqrt{x^2 - 1} \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{x}{\sqrt{x^2 - 1}}dx \\ v = x \end{cases}$

$$\Rightarrow I = x\sqrt{x^2 - 1}\Big|_{\sqrt{2}}^{3} - \int_{\sqrt{2}}^{3} x.\frac{x}{\sqrt{x^2 - 1}}dx = 5\sqrt{2} - \int_{\sqrt{2}}^{3}\left[\sqrt{x^2 - 1} + \frac{1}{\sqrt{x^2 - 1}}\right]dx$$

$$= 5\sqrt{2} - \int_{\sqrt{2}}^{3}\sqrt{x^2 - 1}dx - \int_{\sqrt{2}}^{3}\frac{dx}{\sqrt{x^2 - 1}} = 5\sqrt{2} - I - \ln\left|x + \sqrt{x^2 - 1}\right|\Big|_{\sqrt{2}}^{3}$$

$$\Rightarrow I = \frac{5\sqrt{2}}{2} - \ln\left(\sqrt{2} + 1\right) + \frac{1}{4}\ln 2$$

Chú ý: Không được dùng phép đổi biến $x = \dfrac{1}{\cos t}$ *vì* $\left[\sqrt{2};3\right] \notin \left[-1;1\right]$